MATHEMATICAL REASONING

(KEY CONCEPTS + SOLVED EXAMPLES)

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KEY CONCEPTS

1. Statements

In reasoning we communicate our ideas or thoughts with the help of sentences in a particular language.

"A sentence is called a mathematically acceptable statement if it is either true or false but not both".

A statement is assumed to be either true or false. A true statement is known as a valid statement and a false statement is known as an invalid statement.

Note :

A statement can not be both true and false at the same time.

Note :

(i) Imperative sentences (expresses a request or command), exclamatory sentences (expresses some strong feeling), Interrogative sentences (asks some questions) do not considered as a statement in mathematical language.

(ii) Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements.

2. Truth Table

Truth table is that which gives truth values of compound statements.

It has a number of rows and columns. The number of rows depend upon the number of simple statements.

Note that for n statements, there are 2^n rows.

(i) Truth table for single statement p :

Number of rows = $2^1 = 2$

p
T
F

(ii) Truth table for two statements p and q :

Number of rows = $2^2 = 4$

p	q
T	T
T	F
F	T
F	F

(iii) Truth table for three statements p, q and r.

Number of rows = $2^3 = 8$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

3. Negation of a Statement

The denial of a statement p is called its negation and is written as ~ p, and read as 'not p'.

Negation of any statement p is formed by writing "It is not the case that ...."

or

"It is false that........"

or

inserting the word "not" in p.

4. Compound Statements

If a statement is combination of two or more statements, then it is said to be a compound statement.

And each statement which form a compound statement are known as its sub-statements or component statements.

Basic connectives :

In the compound statement, we have learnt that the words **'or'** & **'and'** connect two or more statements. These are called connectives. When we use these compound statements, it is necessary to understand the role of these words.

4.1 The word "AND" : Any two statements can be connected by the word "and" to form a compound statement.

Rule - (1) The compound statement with word "and" is true if all its component statements are true.

Rule - (2) The compound statement with word "and" is false if any or all of its component statements are false.

Truth table for compound statement with word "And"

The compound statement "p and q" is denoted by "$p \wedge q$".

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

4.2 The word "OR"

Any two statements can be connected by the word "OR" to form a compound statement.

Rule (1) : The compound statement with word "or" is true if any or all of its component statements are true.

Rule (2) : The compound statement with word "or" is false if all its component statement are false.

Truth table for compound statement with word "OR" :

The compound statement "p or q" is denoted by "$p \vee q$".

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4.3 Negation of compound statements

If p and q are two component statements then the negation of the compound statement

(i) ~(p or q) is ~ p and ~ q $\boxed{\sim(p \vee q) = \sim p \wedge \sim q}$

(ii) ~(p and q) is ~ p or ~ q $\boxed{\sim(p \wedge q) = \sim p \vee \sim q}$

5. Some Properties

(1) Prove that : $\sim(\sim p) = p$.

Proof : Truth table

p	~p	~(~p)
T	F	T
F	T	F

We observe that first and third column are identical. Hence $\sim(\sim p) = p$.

(2) Prove that : $\sim(p \wedge q) = \sim p \vee \sim q$

Proof : Truth table

p	q	~ p	~ q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

We observe that last two columns are identical. Hence $\sim(p \vee q) = \sim p \vee \sim q$

(3) Prove that : $\sim(p \vee q) = \sim p \wedge \sim q$

Proof : Truth table

p	q	~ p	~ q	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

We observe that last two columns are identical

Hence $\sim(p \vee q) = \sim p \wedge \sim q$

6. Conditional Statements or Implications

If p and q are any two statement then the compound statement in the form "If p then q" is called a conditional statement or an implication.

The statement "If p then q" is denoted by

$p \to q$ or $p \Rightarrow q$ (to be read as p implies q)

In the implication "$p \to q$", p is called the antecedent (or the hypothesis) and q the consequent (or the conclusion)

Remark :

In the first two statements given above we observe that the hypothesis and conclusion have related subject matters where as in the last two statements do not have related subject matters. In mathematical logic such type of statements are also accepted as a conditional statements.

Truth table for a conditional statement :

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Note :

$p \to q$ is false only when p is true and q is false.

7. Negation and Contrapositive of a Conditional Statement

(1) Negation : If p and q are two statements then

$\sim (p \to q) = p \wedge \sim q$

(2) Contrapositive : If p and q are two statements, then the contrapositive of the implication

$p \to q = (\sim q) \to (\sim p)$

8. Biconditional Statements

If p and q are any two statements then the compound statement in the form of "p if and only if q" is called a biconditional statements and is written in symbolic form $p \leftrightarrow q$ or $p \Leftrightarrow q$

Rule :- $p \leftrightarrow q$ is true only when both p and q have the same value.

p	q	$p \to q$	$q \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Negation of biconditional statement :

$$\sim (p \leftrightarrow q) = (p \wedge \sim q) \vee (\sim p \wedge q)$$

9. Tautology and Fallacy (Contradictions)

(a) Tautology : This is a statement which always true for all truth values of its components.

Ex. Consider $p \vee \sim p$

Truth table

p	~p	$p \vee \sim p$
T	F	T
F	T	T

we observe that last columns is always true. Hence $p \vee \sim p$ is a tautology.

(b) Fallacy (contradiction) : This is statement which is always false for all truth values of its components.

10. Duality

The compound statements s_1 and s_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of each other.

Duality symbol :

Let $s(p, q) = p \wedge q$ be a compound statement

Then $s * (p, q) = p \vee q$ where $s * (p, q)$ is the dual statement of s(p,q).

11. Algebra of Statements

Statements satisfy many laws some of which are given below -

(1) Idempotent Laws : If p is any statement then

(i) $p \vee p \equiv p$

(ii) $p \wedge p \equiv p$

(2) Associative Laws : If p, q, r are any three statements, then

(i) $p \vee (q \vee r) = (p \vee q) \vee r$

(ii) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

(3) Commutative Laws : If p, q are any two statements, then

(i) $p \vee q = q \vee p$ (ii) $p \wedge q = q \wedge p$

(4) Distributive Laws : If p, q, r are any three statements, then

(i) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(5) Identity Laws : If p is any statement, t is tautology and c is a contradiction, then

(i) $p \vee t = t$ (ii) $p \wedge t = p$

(iii) $p \vee c = p$ (iv) $p \wedge c = c$

(6) Complement Laws : If t is a tautology, c is a contradiction and p is any statement, then

(i) $p \vee (\sim p) = t$ (ii) $p \wedge (\sim p) = c$

(iii) $\sim t = c$ (iv) $\sim c = t$

(7) Involution law : If p is any statement, then $\sim(\sim p) = p$

(8) De morgan's law : If p and q are two statements, then

(i) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

(ii) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

SOLVED EXAMPLE

Ex.1 Which of the following is not a statement -

(A) $\sqrt{3}$ is a rational number

(B) 8 > 7

(C) Please open the door

(D) All prime numbers are odd.

Sol. **Ans. [C]**

Ex.2 Negation of '3' is an odd number and 7 is a rational number is -

(A) 3 is not an odd number and 7 is not a rational number

(B) 3 is an odd number or 7 is a rational number

(C) 3 is an odd number or 7 is not a rational number

(D) 3 is not an odd number or 7 is not a rational number

Sol. use the property

$\sim (p \text{ and } q) = \sim p \text{ or } \sim q$ **Ans.[D]**

Ex.3 The negation of statement

$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$ is -

(A) $(p \vee \sim q) \wedge (p \vee q)$

(B) $(p \wedge \sim q) \vee (p \vee q)$

(C) $(\sim p \vee q) \vee (\sim p \wedge \sim q)$

(D) $(p \wedge \sim q) \wedge (p \vee q)$

Sol. Use the property

$\sim (a \wedge b) = \sim a \vee \sim b$ **Ans.[B]**

Ex.4 The negation of statement

$(p \wedge q) \vee (q \vee \sim r)$

(A) $(p \wedge q) \vee (\sim q \vee \sim r)$

(B) $(\sim p \wedge \sim q) \wedge (\sim q \wedge r)$

(C) $(\sim p \vee \sim q) \wedge (\sim q \wedge r)$

(D) None of these

Sol. use the property

$\sim (a \vee b) = \sim a \wedge \sim b$ **Ans.[C]**

Ex.5 The statement $(p \wedge \sim q) \vee p$ is logically equivalent to -

(A) p (B) ~p

(C) q (D) ~q

Sol.

p	q	~q	$p \wedge \sim q$	$(p \wedge \sim q) \vee p$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	F
F	F	T	F	F

Hence statement $(p \wedge \sim q) \vee p$ is logically equal to statement p **Ans.[A]**

Ex.6 If $(p \wedge \sim q) \vee (q \wedge r)$ is true and. q and r both true then p is -

(A) True

(B) False

(C) may be true or false

(D) none

Sol. Let p be true then statement

$(p \wedge \sim q) \vee (q \wedge r) = (T \wedge F) \vee (T \wedge T)$

$= F \vee T$

$= T.$

also let p be false then statement

$(p \wedge \sim q) \vee (q \wedge r) = (F \wedge F) \vee (T \wedge T)$

$= F \vee T$

$= T.$

$\therefore$ p may be true or false. **Ans.[C]**

Ex.7 Negation of the statement

If a number is prime then it is odd' is.

(A) A number is not prime but odd.

(B) A number is prime and it is not odd.

(C) A number is neither primes nor odd.

(D) None of these

Sol. use the property

$\sim (p \rightarrow q) = p \wedge \sim q$

Hence (B) is correct option. **Ans.[B]**

Ex.8 If p, q, r are substatements with truth values. T, T, F then

the Statement $r \rightarrow (p \wedge \sim q) \vee (\sim q \wedge \sim r)$ will be

(A) True

(B) False

(C) may be true or false

(D) None of these

Sol. $F \to (T \wedge F) \vee (F \wedge T)$

$F \to (F \vee F)$

$F \to F = T$ **Ans.[A]**

Ex.9 The Negation of the statement $(p \wedge q) \to r$ is -

(A) $(\sim p \vee \sim q) \to r$

(B) $(\sim p \wedge \sim q) \wedge \sim r$

(C) $(p \wedge q) \wedge \sim r$

(D) $(\sim p \vee \sim q) \wedge r$

Sol. use the property

$\sim (a \to b) = a \wedge \sim b$

Hence (C) is correct option **Ans.[C]**

Ex.10 The contrapositive of $p \Rightarrow (\sim p \wedge q)$ is -

(A) $\sim p \Rightarrow (p \vee \sim q)$

(B) $(p \wedge \sim q) \Rightarrow \sim p$

(C) $(p \vee \sim q) \Rightarrow p$

(D) $(p \vee \sim q) \Rightarrow \sim p$

Sol. contrapositive of $a \Rightarrow b$ is $\sim b \Rightarrow \sim a$

Hence (D) is correct option **Ans.[D]**

Ex.11 $(\sim p \vee q)$ is logically equal to -

(A) $p \to q$ (B) $q \to p$

(C) $\sim (p \to q)$ (D) $\sim (q \to p)$

Sol. $\sim (p \to q) = p \wedge \sim q$

$p \to q = \sim p \vee q$

Hence (A) is correct option **Ans.[A]**

Ex.12 The statement $p \Leftrightarrow q$ is equal to -

(A) $(\sim p \vee q) \vee (p \vee q)$

(B) $(p \wedge q) \vee (\sim p \wedge \sim q)$

(C) $(\sim p \vee q) \wedge (p \vee \sim q)$

(D) $(p \wedge q) \vee (p \vee q)$

Sol. We know that

$\sim (p \Leftrightarrow q) = (p \wedge \sim q) \vee (\sim p \wedge q)$

$(p \Leftrightarrow q) = (\sim p \vee q) \wedge (p \vee \sim q)$

Hence (C) is correct option **Ans.[C]**

Ex.13 The statement $(p \wedge q) \Leftrightarrow \sim p$ is a

(A) Tautology

(B) contradiction

(C) Neither tautology nor contradiction

(D) None of these

Sol.

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \Leftrightarrow \sim p$
T	T	T	F	F
T	F	F	F	T
F	T	F	T	F
F	F	F	T	F

Hence the statement neither tautology nor contradiction.

Ans.[C]

Ex.14 The statement $p \to p \vee q$ is a -

(A) Tautology

(B) Contradiction

(C) Neither tautology nor contradiction

(D) None of these

Sol.

p	q	$p \vee q$	$p \to p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Hence the statement is a tautology.

Ans.[A]

Ex.15 The statement $(p \to \sim q) \Leftrightarrow (p \wedge q)$ is a -

(A) Tautology

(B) contradiction

(C) Neither tautology nor contradiction

(D) None

Sol.

p	q	$\sim q$	$p \to \sim q$	$p \wedge q$	$(p \to \sim q) \Leftrightarrow p \wedge q$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

Hence the statement is a contradiction

Ans.[B]